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TWO - SECANT FORMULA

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Two - Secant Formula

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We give here a proof of the well-known "secant-formula", (see [S-R],p.214). There exist several proofs of this formula, in [H], [Kl], [LK], [P-S]. Our proof, based on the existence of a secant-scheme, as defined in [La], shows that the formula actually gives the number of 2-secants of X in $P = \mathbb{P}_k^{2\dim X+1}$ through a general point of P, multiplicities counted. Furthermore, one would hope that it prepares the way for a similar study of r-secants, $r \geq 3$.

I. STRATIFICATION OF THE SECANT-SCHEME.

Let X be a closed algebraic subvariety of a projective space $P = \mathbb{P}_k^N$ over an algebraically closed field k.

Definition: Let r be a positive integer. A line l in P is said to be an r-secant of X if $X \times_P l$ (the intersection of l and X in P) is a 0-dimensional scheme of degree r.

Let us denote by G the Grassmann scheme parametrizing the rank 2 quotients of k^{N+1} (or equivalently lines in P) and by Q the universal rank 2 quotient bundle of k_G^{N+1} on G. In [La], Laudal defined a locally closed subscheme $\mathrm{Sec}_r(X)$ of G, the r-secant scheme of X, parametrizing the r-secants of X. Let us recall briefly his definition.

Consider the following fiber square, where $L = \mathbb{P}_G(Q)$ denotes the tautological line bundle on G, naturally embedded in $P \times G$, and $Z = L \underset{P\times G}{\times} (X\times G)$

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, "t"] & L \arrow[d] \\ X\times G \arrow[r, "(i,1)"] & P\times G \arrow[r, "p_2"] & G \end{tikzcd}$$

(p_i, $i=1,2$, will always denote the projection of a product on the first and second factor respectively).

As a coherent sheaf on $P \times G$, the structure sheaf O_Z of Z defines a "flattening stratification of G"([M], lecture 8), i.e. a disjoint family of locally-closed subschemes Y_i of G such that $G = \bigcup_{i=1}^{n} Y_i$ and $O_{Z \times_G Y_i}$ is flat over Y_i. Let $\mathrm{Sec}_r(X)$, <u>the r-secant scheme of X</u>, denote the union of all Y_i such that the fibers of $p_2 \circ t$ over Y_i are 0-dimensional of degree r. (The Y_i being indexed by the Hilbert-polynomial of $O_{Z\times_G Y_i}$ on $P \times Y_i$.)

$\mathrm{Sec}_r(X)$ is a locally closed subscheme of G, the k-points of which actually correspond to the r-secants of X.

$\mathrm{Sec}_r(X)$ represents a subfunctor $\underline{\mathrm{Sec}}_r(X)$ of the Grassmann functor $\underline{G}$. One may prove (see [La]) that the fiber functors of $\underline{\mathrm{Sec}}_r(X)$ are deformation functors, the hull of which $\hat{O}_{\mathrm{Sec}_r(X),l}$ (the completed local ring of $\mathrm{Sec}_r(X)$ at l) can be computed. This provides local information on $\mathrm{Sec}_r(X)$. One result says that if an r-secant l is intersecting X in r distinct non singular points of X, l is a non singular point of $\mathrm{Sec}_r(X)$ if and only if the embedding-dimension of $O_{\mathrm{sec}_r(X),l}$ is equal to $2(N-1)-r(N-\dim X-1)$. For $r \leq 2$, $\mathrm{Sec}_r(X)$ is always non singular at l.

II. TWO-SECANT FORMULA

Let $X \overset{i}{\to} P$ be a closed embedding as above, X being non singular. Writing $S_2 = \mathrm{Sec}_2(X)$, let L_2 be the restriction to S_2 of the tautological bundle L on G, let $s_2 : L_2 \to P \times S_2$ be the canonical embedding and $\varphi_2 = p_1 \circ s_2 : L_2 \to P$.

THEOREM: Suppose $X \overset{i}{\to} P$ is such that through a generic point of P there passes a finite number $\neq 0$ of 2-secants. Then

i) $N = 2 \dim X+1$

$\dim S_2 = 2 \dim X$ and

$\overline{\mathrm{im}\ \varphi_2} = P$.

ii) Moreover S_2 is non singular and φ_2 is generically quasi-finite of degree δ_2, the number of 2-secants through the generic point of P.

iii) δ_2 is given by the following two-secant formula:

$$2\delta_2 = \deg^2(X) - \sum_{i=0}^{n} \binom{2n+1}{n+i} \deg s_i(X)$$

where $s_i(X)$ is the i^{th} Segre-class of X as an element of the Chow-ring $A^{\circ}(X)$ and $\deg(\ )$ denotes the degree in P of a subscheme or its corresponding class in $A^{\circ}(P)$.

Let us prove the theorem.

1. Proof of i) and ii).

As an easy consequence of [La] one deduces that S_2, if non empty, is non singular of dimension $2 \dim X$. In fact, if $l \in S_2$ is not a tangent, S_2 is non singular at l, of dimension $2 \dim X$. This follows from $\dim_k A^1 = 2 \dim X = 2(N-1) - 2(N-d-1) + \varepsilon$ and $\dim_k A^2 = \dim Q = 0$ (see the discussion preceding the "trisecant lemma", §3 loc. cit.)
If $l \in S_2$ is a tangent, we still find $\dim_k A^1 = 2 \dim X$, $A^2 = 0$ (see the forthcoming thesis of Tore Wentzel-Larsen), so that S_2

is non singular at l. In fact (see §3 loc. cit.)

$$A^1 = \ker(l, m-n)$$
$$A^2 = \mathrm{Coker}\,(l, m-n)$$

where l, m and n may be computed as follows. Suppose X is defined by the homogeneous ideal $\underline{a}$ of $k[X_o,\dots,X_N]$ and l by the ideal $(X_2,\dots,X_N)$. Let x be the point in which l cuts X. We may assume $\underline{a}\ O_{X,x} = (f_{d+1},\dots,f_N)$, where $d = \dim X$, $x = (1,0,\dots,0)$. Let $Y_i = \dfrac{X_i}{X_o}$, $i=1,\dots,N$. The morphisms l, m, n of the diagram of ([La$_2$]§3) is:

$$\begin{tikzcd}[column sep=tiny]
H^o(X,N_X) \arrow[dr, "o"'] & & \mathrm{Hom}_{k[\varepsilon]}(A/A^2,k[\varepsilon]) \arrow[dl, "l"'] \arrow[dr, "m"] & & \mathrm{Hom}_{k[X_0,X_1]}\left(\mathrm{gr}\dfrac{(X_2,\dots X_N)}{(X_2,\dots X_N)^2},k[X_0,X_1]\right)_0 \arrow[dl, "n"] \\
& \mathrm{Hom}_{k[\varepsilon]}\left(\dfrac{(f_{d+1},\dots f_N)}{A(f_{d+1},\dots f_N)},k[\varepsilon]\right) & & \mathrm{Hom}_{k[\varepsilon]}\left(\dfrac{(Y_2,\dots Y_N)}{(Y_2,\dots,Y_N)^2},k[\varepsilon]\right) &
\end{tikzcd}$$

where $A = (Y_1^2, Y_2,\dots Y_N)$ is the ideal of $z = X \cap l$ in the affine piece where $X_o \neq 0$ and $k[\varepsilon] = k[Y_1,\dots Y_N]/A$. Consider first A^1; clearly n is an isomorphism, thus $A^1 = \ker l$. Since X is non singular, $\mathrm{rank}_{k[\varepsilon]}((f_{d+1},\dots f_N)/A(f_{d+1},\dots f_N),k[\varepsilon]) = N-d$. By a proper choice of bases in the diagram above, one gets $A^1 = \ker M$, where M is the matrix

$$\begin{pmatrix} M_1 & 0 \\ M_2 & M_1 \end{pmatrix} \quad \text{where} \quad M_1 = \begin{pmatrix} & \vdots & \vdots & & \vdots \\ & \dfrac{1}{2}\dfrac{\partial^2 f_i(x)}{\partial Y_1^2} & \dfrac{\partial f_i(x)}{\partial Y_2} & \cdots & \dfrac{\partial f_i(x)}{\partial Y_N} \\ & \vdots & \vdots & & \vdots \end{pmatrix}$$

and M_2 contains higher derivatives of the f_i's. By non singularity of X, rank M = 2rank $M_1 = 2(N-d)$, therefore coker $(l,m-n) =$ coker $l = 0$, i.e. $A^2 = 0$.

Since S_2 is non singular of dimension $2\dim X$, we have $\dim L_2 = 2\dim X + 1$. By assumption, the generic fiber of φ_2 is finite and non empty, which implies that $\dim L_2 = \dim P$, that is, $N = 2\dim X + 1$ and $\overline{\operatorname{im}\varphi_2} = P$. φ_2 is generically quasi-finite of degree δ_2, the number of 2-secants through the generic point of P. In order to compute δ_2, we need a useful morphism $\sigma : X \tilde{\times} X \to G$, where $X \tilde{\times} X$ denotes the blowing up of $X \times X$ along the diagonal.

2. <u>The morphism $\sigma : X \tilde{\times} X \to G$.</u>

Let $P \tilde{\times} P$ denote the blowing-up of $P \times P$ along the diagonal, and ε_P (resp. ε_X) the exceptional locus in $P \tilde{\times} P$ (resp. $X \tilde{\times} X$). Then there exists a morphism $\gamma : P \tilde{\times} P \to G$ (see [Kl.],V.B). Let us recall the definition.

a) If Ω_P^1 denotes the cotangent sheaf on P, we have the following exact sequence on P.

$$0 \to \Omega_P^1(1) \to 0_P^{N+1} \to 0_P(1) \to 0.$$

Let R denote the projective bundle $\mathbb{P}_P(\Omega_P^1(1)$ and p the structure morphism $R \xrightarrow{p} P$. Let K_R be the kernel of the natural surjective homomorphism $p^* \Omega_P(1) \to 0_R(1) \to 0$, and E the locally free rank 2 0_R-module defined to make the following diagram exact and commutative:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K_R \arrow[r, equal] \arrow[d] & K_R \arrow[d] & & \\
0 \arrow[r] & p^*\Omega_P^1(1) \arrow[r] \arrow[d] & 0_R^{N+1} \arrow[r] \arrow[d] & p^*0_P(1) \arrow[r] \arrow[d, equal] & o. \\
0 \arrow[r] & 0_R(1) \arrow[r, dashed] \arrow[d] & E \arrow[r, dashed] \arrow[d] & p^*0_P(1) \arrow[r] & o. \\
 & 0 & 0 & &
\end{tikzcd}$$

E, as a rank 2 quotient of 0_R^{N+1}, provides a morphism

$\alpha : R \to G$ such that $\alpha^* Q = E$. The fiber of p over a point $x \in P$ consists, as a set, of all lines in P through x. α takes a point of R, that is, a line l through $x \in P$, to the corresponding point l in G. The factorization $O_R^{N+1} \to \alpha^* Q = E \to p^* O_P(1)$, where all the homomorphisms are surjective, gives a factorization of (p,α) through $\mathbb{P}_G(Q)$:

$$\begin{tikzcd} R \arrow[rr, "{(p,\alpha)}"] \arrow[dr] & & P \times G \\ & \mathbb{P}_G(Q) \arrow[ur] & \end{tikzcd}$$

and it is easy to prove that the morphism $R \to \mathbb{P}_G(Q)$ defined by $E \to p^* O_P(1) \to 0$ is in fact an isomorphism, by which we can identify R and $\mathbb{P}_G(Q)$ as $(P \times G)$-schemes.

b) There is on the other hand the key morphism $\lambda : P \tilde{\times} P \to R$, defined in [Kl.]. (See also [H], [Lk]). λ can be described as follows. Outside of ε_P, λ is equal to the projection of $P \times P$ from the diagonal $\mathbb{P}_P(O_P(1))$ on $R = \mathbb{P}_P(\;{}^1_P(1))$. λ takes a point (x,y) not in ε_P to $(x,\overline{xy}) \in R \subset P \times G$, where $\overline{xy}$ is the line through x and y as an element of G, while the restriction of λ to $\varepsilon_P = \mathbb{P}_P(\Omega^1_P)$ is the natural P-isomorphism $\mathbb{P}_P(\Omega^1_P) \simeq \mathbb{P}_P(\Omega^1_P(1))$. Consider now the projective bundle $\mathbb{P}_R(E)$ over R. It is a subbundle of $P \times R$ and a $(P \times P)$-sceme by the composed morphism

$$\begin{tikzcd} \mathbb{P}_R(E) \arrow[r] \arrow[dr] & P \times P \arrow[d, "{(p_1,p)}"] \\ & P \times P \end{tikzcd}$$

$P \tilde{\times} P$ can be identified with $\mathbb{P}_R(E)$ both as R-scheme (via λ) and a $P \times P$-scheme (via the blowing-up morphism).

c) Consider the morphism $\gamma = \alpha \circ \lambda : P \tilde{\times} P = \mathbb{P}_R(Q_L) \xrightarrow{\lambda} R = \mathbb{P}_G(Q) = L \xrightarrow{\alpha} G$. By the identification $\mathbb{P}_R(\alpha^*Q) \simeq \mathbb{P}_G(Q) \times_G \mathbb{P}_G(Q) = L \times_G L$, the subscheme $\mathbb{P}_R(O_R(1))$ of $\mathbb{P}_R(Q_L)$ is identified with the diagonal of $L \times_G L$, the morphism λ with the first projection, and γ with the structure morphism $L \times_G L \to G$. By the inclusion $L \subset P \times G$, $L \times_G L$ has a natural structure of $P \times P$-scheme and the identification with $P_R(\alpha^*Q)$, whence with $P \tilde{\times} P$, is a $P \times P$-isomorphism. The morphism $\gamma : P \tilde{\times} P \to G$ is "natural" in the following sense: it takes a point of $P \tilde{\times} P = L \times_G L$, that is, a triple (x,y,l) where l is a line through $x, y \in P$, to the corresponding point $l \in G$.

d) $X \tilde{\times} X$ can be seen as the proper transform of $X \times X$ under the blowing up $P \tilde{\times} P$. We define $\sigma : X \tilde{\times} X \to G$ to be the restriction of γ to $X \tilde{\times} X$.

3. <u>Proof of the secant-formula iii)</u>.

Let S denote the scheme-theoretical image by σ of $X \tilde{\times} X$ in G. Since X is reduced, irreducible, so are $X \tilde{\times} X$ and S. We have clearly $S_2 \subset S$. Since S_2 is non singular and $\dim S_2 = \dim X \tilde{\times} X = 2 \dim X$, S is the closure in G of S_2. Let L_S denote the restriction of L to S and φ_S the composed morphism of the natural embedding $L_S \to P \times S$ with p_1

$$\begin{array}{ccc} L_S & \longrightarrow & P \times S \\ & \searrow^{\varphi_S} & \downarrow p_1 \\ & & P \end{array}$$

We have clearly $S = \bar{S}_2$, $L = \bar{L}_2$, $\operatorname{im} \varphi_S = \overline{\operatorname{im} \varphi_2}$ and φ_S is generically finite of degree δ_2.

Let $B = X \tilde{\times} X$, let Q_B denote the pull-back of Q to B, with $L_B = \mathbb{P}_B(Q_B)$ and let σ_L be the induced morphism $L_B \to L_S$

in the cartesian diagram

$$\begin{tikzcd} L_B \arrow[r, "\sigma_L"] \arrow[d] \arrow[dr, phantom, "\square"] & L_S \arrow[r] \arrow[d] \arrow[dr, "\varphi_S"'] & P \times S \arrow[d, "p_1"] \\ B \arrow[r, "\sigma"] & S & P \end{tikzcd}$$

σ, and hence σ_L, are generically finite of degree 2.
Put $\varphi = \varphi_S \circ \sigma_L$. Then φ is generically of finite degree:

$$\deg \varphi = 2 \deg \varphi_S = 2\delta_2 .$$

Let us compute $\deg \varphi$.
We denote by $\tilde{p}_i$. $P \tilde{\times} P \to P$, $i=1,2$, the composed morphism of the blowing up $P \tilde{\times} P \to P \times P$ and the projection p_i, $i=1,2$, as well as its restriction to $X \tilde{\times} X$ if no confusion is possible. Consider the following commutative diagram:

$$\begin{tikzcd} & B \arrow[rr, "j"] \arrow[d, "r"] \arrow[drr, "t"] \arrow[ddl, "\tilde{p}_1"'] & & L_B \arrow[d, "s"] \arrow[ddl, "\varphi"'] & \\ & X \times B \arrow[rr, "{(i,1)}"] \arrow[dl, "p_1"] & & P \times B \arrow[dl, "p_1"] \arrow[dr, "p_2"] & \\ X \arrow[rr, "i"] & & P & & B \end{tikzcd} \qquad (*)$$

where $r = (\tilde{p}_1, 1)$, $t = (i,1) \circ r$, and s is the natural embedding $L_B \to P \times B$. The image by t of an element $b \in B$ lying over $(x,y) \in X \times X$ is (x,b). We have seen that b corresponds to a line through x and y ; we have therefore $(x,b) \in L_B$ and t factorizes through L_B. Let j be such that $t = s \circ j$.

All the schemes considered are quasi-projective, non singular. We can therefore identify their Chow cohomology and homology groups of cycles modulo rational equivalence.

To the proper morphism φ corresponds a homomorphism φ_* of Chow cohomology groups, $\varphi_* : A^\bullet(L_B) \to A^\bullet(P) = \mathbb{Z}[T]/T^{n+1}$ where T is the class in $A^1(P)$ of an hyperplane. L_B is reduced, irreducible, and φ has generic degree $2\delta_2$, so we have

$$\varphi_*(1) = 2\delta_2 \ .$$

a) Let us compute first $s_*(1)$, the class of L_B in $A^\bullet(P\times B)$.

Let K denote the kernel of the quotient $V_B \to Q_B \to o$ defined by σ, with $V = k^{N+1}$. By [Lk] (lemma 2 p.173), $L_B = \mathbb{P}_G(Q_B)$ is the "scheme of zeroes" of the homomorphism $\kappa : K_{P\times B} \to O_{P\times B}(1)$ obtained by composing the homomorphism $o \to K_{P\times B} \to V_{P\times B}$ with the surjective homomorphism $V_{P\times B} \to O_{P\times B}(1)$ on $P \times B = \mathbb{P}_B(V)$.

$P\times B$ is non singular and L_B has codimension $N-1$, equal to the rank of $K_{P\times B}$. Let us denote by $c_i(F)$ the i^{th} Chern class in $A^i(P\times B)$ of a locally free $O_{P\times B}$-module F, and by $s_i(F)$ the i^{th} "inverse" Chern class of F in $A^i(P\times B)$, defined by $\left(\sum_{i\in\mathbb{N}} s_i(F)\right)\left(\sum_{i\in\mathbb{N}} c_i(F)\right) = 1$. Then by [G] (théorème 2) the class $[L_B]$ of L_B in $A^\bullet(P\times B)$ is given by

$$(1) \qquad [L_B] = c_{N-1}(\check{K}_{P\times B} \otimes_{O_{P\times B}} O_{P\times B}(1))$$

where $\check{K}_{P\times B}$ is the dual $O_{P\times B}$-module of $K_{P\times B}$.

b) Let us prove

$$(2) \qquad \varphi_*(1) = p_{1*}[L_B] = p_{1*}(s_{N-1}(Q_{P\times B})) \ .$$

By the exact sequence $o \to K_{P\times B} \to V_{P\times B} \to Q_{P\times B} \to o$, we get

$$c_{N-1}(\check{K}_{P\times S} \otimes O_{P\times S}(1)) = \sum_{i\geq o} (\check{Q}_{P\times B})T^{N-1-i} \ .$$

But $p_{1*}(s_i(\check{Q}_{P\times B})\, T^{N-1-i}) = T^{N-1-i} p_{1*} s_i(\check{Q}_{P\times B})$ and since $\dim B = N-1$, $p_{1*}(s_i(\check{Q}_{P\times B}) = o$ for all $i < N-1$. We note that, $(N-1)$ being even, $s_{N-1}(Q_{P\times B}) = s_{N-1}(\check{Q}_{P\times B})$.

Q.E.D.

c) Consider the diagram (*) above.

We have in $A^{\bullet}(P)$ the following formula:

(3) $$p_{1*}t_*(s_{N-1}Q_B) = \delta_2 T^N \ .$$

Proof of (3). The morphism $p_2 \circ t$ is the identity morphism on B.
Let us write

$$t_*(1) = \sum_{i\geq o} b_i T^{N-i}, \quad b_i \in A^i(B).$$

Then

$$(p_2 \circ t)_*(1) = id_{B*}(1) = 1$$

and

$$p_{2*}(t_*(1)) = p_{2*}(b_o T^N) = b_o$$

which implies $b_o = 1$ in $A^{\bullet}(B)$.

By functioriality of the Chern classes, we have:

$$s_{N-1}(Q_B) = t^* s_{N-1}(Q_{P\times B}).$$

Then (2) implies (3), by:

$$p_{1*}t_*(s_{N-1}(Q_B)) = p_{1*}(s_{N-1}(Q_{P\times B})t_*(1)) \quad \text{(projection formula)}$$

$$= p_{1*}(s_{N-1}(Q_{P\times B})\sum_{i=o}^{N-1} b_i T^{N-i}) = p_{1*}(s_{N-1}(Q_{P\times B}) b_o T^N)$$

$$= T^N p_{1*}(s_{N-1}(Q_{P\times B})) = \delta_2 T^N \ .$$

d) In the commutative diagram (*) , we have:

$p_1 \circ t = i \circ p_1 \circ r = i \circ \tilde{p}_1$, so we can replace (3) by (3'):

$$\delta_2 T^N = i_* \tilde{p}_{1*}(s_{N-1}Q_B) \ .$$

In order to compute $s_{N-1}(Q_B)$, let us put

$$\alpha_1 = \tilde{p}_1^{\,*} i^*(T)$$

$$\alpha_2 = \tilde{p}_2^{\,*} i^*(T)$$

and let τ denote the inclusion morphism $\varepsilon_X \to B$ of the exceptional locus ε_X of B.

Then we have in $A^{\cdot}(B)$:

<u>Lemma 2</u>: $c_1(Q_B) = \alpha_1 + \alpha_2 - \tau_*(1)$

$$c_2(Q_B) = \alpha_1 + \alpha_2 - \tau_*(1) \ .$$

<u>Proof of lemma 2:</u> We have identified $P \tilde{\times} P$ with $L \times_G L$ as G-schemes and $P \times P$ -schemes. Let $\tilde{i}$ denote the embedding

$$X \tilde{\times} X \xrightarrow{\tilde{i}} L \times_G L = P \tilde{\times} P$$

which corresponds to $i : X \to P$ in a natural way. Let Δ denote the diagonal in $L \times_G L$ and

$$\mathcal{L}_1 = p_1^{\,*} \mathcal{O}_L(1)$$

$$\mathcal{L}_2 = p_2^{\,*} \mathcal{O}_L(2) \ .$$

Look at the morphisms λ and α :

$$L \times_G L = \mathbb{P}_L(Q_L) \xrightarrow{\lambda} L = \mathbb{P}_G(Q) \xrightarrow{\alpha} G.$$

By identifying $L \times_G L$ and $\mathbb{P}_L(Q_L)$ we also get:

$$\Delta = \mathbb{P}_L(\mathcal{O}_L(1)) \ , \quad \mathcal{L}_1 = \lambda^* \mathcal{O}_L(1) \ ,$$

$$\mathcal{L}_2 = \mathcal{O}_{\mathbb{P}_L(Q_L)}(1).$$

Using again [Lk] (lemma 2), we see that the L-subbundle Δ of $L \times_G L$ is the "scheme of zeroes" of the composed morphism:

$\lambda^* H \to Q_{\mathbb{P}_L(Q_L)} \to Q_{\mathbb{P}_L(Q_L)}(1)$, where H is the kernel of $Q_L \to O_L(1) \to o$ on L, and by [G] (théorème 2) we get

$$[\Delta] = c_1(\lambda^* \check{H} \otimes Q_{\mathbb{P}_L(Q_L)}(1)) \quad \text{in} \quad A^{\cdot}(\mathbb{P}_L(Q_L))$$
$$= - c_1(\lambda^* H) + c_1(\mathcal{L}_2).$$

By the exact sequence $o \to H \to Q_L \to O_L(1) \to o$ this becomes

$$[\Delta] = - c_1(Q_{L \underset{G}{\times} L}) + c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2).$$

Since $\mathcal{L}_1$ is a quotient of Q, we get

$$c_1(Q_{L \underset{G}{\times} L}) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2) - [\Delta]$$
$$c_2(Q_{L \underset{G}{\times} L}) = c_1(\mathcal{L}_1)\ (c_1(\mathcal{L}_2) - [\Delta]\ .$$

To prove the lemma, it is enough to see that $\tilde{i}^*[\Delta] = \tau_*(1)$. By $P \tilde{\times} P = L \underset{G}{\times} L$, Δ is equal to the exceptional divisor ε_P and we know that the restriction of the invertible sheaf $O_{P\tilde{\times}P}(\varepsilon_P)$ to $X \tilde{\times} X$ is $O_{X\tilde{\times}X}(\varepsilon_X)$, which implies that

$$\tilde{i}^*[\Delta] = [\varepsilon_X] = \tau_*(1).$$

Q.E.D.

It is now easy to compute $s_{N-1}(Q_B) = s_{2n}(Q_B)$ where $n = \dim X$. By lemma 2

$$s_{2n}(Q_B) = \sum_{i=o}^{2n} \alpha_1^i(\alpha_2 - \tau_*(1))^{2n-i}\ .$$

Clearly $\alpha_1^i\ \alpha_2^{2n-i} = o$ for all $i \neq n$, so we get:

$$s_{2n}(Q_B) = \alpha_1^n\ \alpha_2^n + \tau_*(1) \sum_{i=o}^{2n}\ \alpha_1^{2n-i} \sum_{j=1}^{i} (-1)^j \binom{i}{j} \alpha_2^{i-j} \tau_*(1)^{j-1}\ .$$

But $\tau^{*}\tau_{*}(1) = c_1(N_{\varepsilon_X|B})$ (self intersection formula)

where $N_{\varepsilon_X|B}$ is the conormal sheaf of ε_X in B and we know that on $\varepsilon_X = \mathbb{P}_X(\Omega_X^1)$, $N_{\varepsilon_X|B} = \mathcal{O}_{\mathbb{P}_X(\Omega_X^1)}(1)$ (Ω_X^1 denotes the cotangent sheaf on X). Let us put $\xi = \tau^{*}\tau_{*}(1) = c_1(\mathcal{O}_{\mathbb{P}_X(\Omega_X^1)}(1))$. Then, using the "projection formula", we get

$$s_{2n}(Q_B) = \alpha_1^n\,\alpha_2^n + \tau_{*}\left(\sum_{i=1}^{2n}\tau^{*}(\alpha_1)^{2n-i}\sum_{j=1}^{i}(-1)^{2j-1}\binom{i}{j}\tau^{*}\alpha_2^{i-j}\xi^{j-1}\right)$$

Let e be the structure morphism $\varepsilon_X \xrightarrow{e} X$, and diag the diagonal morphism in the following diagram:

$$\begin{tikzcd}
\varepsilon_X = \mathbb{P}_X(\Omega_X^1) \arrow[r, "\tau"] \arrow[d, "e"] & X\tilde{\times}X \arrow[d] & \\
X \arrow[r, "\mathrm{diag}"] \arrow[dr, equal] & X\times X \arrow[r, "(i,i)"] \arrow[d, "p_1"] & P\times P \arrow[d, "p_1"] \\
& X \arrow[r] & \underline{P}
\end{tikzcd}$$

It is easily seen that

$$\tau^{*}\alpha_1 = \tau^{*}\alpha_2 = e^{*}i^{*}(T).$$

Consider formula (3').

$$i_{*}\tilde{p}_{1*}s_{2n}(Q_B) = i_{*}\tilde{p}_{1*}(\alpha_1^n\alpha_2^n) - i_{*}e_{*}\Big(\sum_{i=1}^{2n}\sum_{j=1}^{i}e^{*}i^{*}(T)^{2n-j}\binom{i}{j}\xi^{j}\quad .$$

Clearly $i_{*}\tilde{p}_{1*}(\alpha_1^n\alpha_2^n) = p_{1*}(i,i)_{*}(i^{*}T^n \otimes i^{*}T^n)$

$$= p_{1*}(T^n i_{*}(1) \otimes T^n i_{*}(1)) = p_{1*}(T^{2n+1}\deg X \otimes T^{2n+1}\deg X)$$

$$= (\deg X)^2\, T^{2n+1}$$, and

$$i_*\tilde{p}_{1*}s_{2n}(Q_B) = (\deg X)^2 T^{2n+1} - \sum_{j=n}^{2n} \sum_{i=j}^{2n} \binom{i}{j} T^{2n-j} i_* e_*(\xi^{j-1}) .$$

Let s_i denote the i^{th} Segre class of X.
By definition, $e_*(\xi^{j-1}) = s_{j-n}(X)$, and by the identity $\sum_{i=j}^{2n} \binom{i}{j} = \binom{2n+1}{j}$ we get

$$i_* e_* s_{2n}(Q_B) = (\deg X)^2 \ T^{2n+1} - \sum_{j=o}^{n} \binom{2n+1}{n+j} \ i_* s_j(X) T^{2n-j} .$$

Finally, this implies the two-secant formula:

$$2\delta_2 = (\deg X)^2 - \sum_{j=o}^{n} \binom{2n+1}{n+j} \deg(s_j(X)).$$

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